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Mode and Energy Guidance Properties of a Slab of Inhomogeneous Medium with Transverse Variations of the Gain Only

LAURA RONCHI ABBOZZO AND RICCARDO PRATESI

***Abstract*—The mode and energy guidance properties of a planar slab of parabolic graded index medium are examined when there are transverse variations of the gain or of the losses only.**

Mode configurations and propagation constants are evaluated of the first four even modes. The results are presented and discussed. In particular it is found that a gain decreasing away from the symmetry plane does not favor the existence of guided modes, as happens when the graded index medium is not limited to a slab. Evidence is found that the presence of the boundaries affects the mode propagation even when the caustic surface is well inside the slab.

I. Introduction

THE PURPOSE of the analysis described in the present paper is to study the mode guidance and the energy guidance properties, at optical frequencies, of a planar slab of graded index medium where there are transverse variations of the gain or of the losses only.

The possibility of mode guidance of an infinitely extended graded index medium (not limited to a slab) based on the transverse variations of the imaginary part of its refractive index has been extensively discussed by Marcuse [1]. In particular, Marcuse has shown that losses increasing or gain decreasing away from the symmetry plane $x=0$ may allow mode guidance even in "inverted" media, namely in media where the real part of the refractive index $\mathrm{Re}\, n(x)$ is an increasing function of the distance $|x|$ from the symmetry plane. By the term guided mode a beam is intended whose amplitude decays exponentially when $|x| \to \infty$. In the opposite case, the mode is termed "leaky".

When the graded index medium is limited to a slab, the situation appears different, at least in the ranges of parameters we have considered. For example, a gain decreasing away from the symmetry plane turns out not to favor the existence of guided modes (Section II). On the contrary, guided modes are found when the gain increases outwards. Another example is that the field amplitude distribution inside the slab turns out to be practically independent of the value of the parameter describing the transverse variations of the gain of the medium (Section III). These results seem to indicate that the mechanism of mode guidance of a slab is substantially different from that operating in the infinitely extended medium, and consequently, that the conclusions valid for the infinite medium cannot be simply extended to the case of a slab.

It is often suggested that, if a mode of the infinitely extended medium (whose amplitude has a Gaussian distribution, as is well known [1]) is sufficiently small where the slab has its boundaries, the presence of the boundaries does not substantially affect the mode distribution itself. This may be true, of course, but the question arises how

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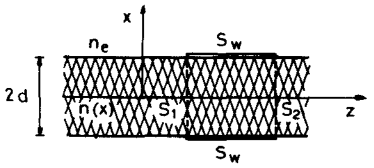

Fig. 1. A section of a slab of graded index medium

small has the field to be at the boundaries, or how small has the index discontinuity at the boundaries to be, in order that the mode distribution inside the slab will not be affected.

The criterion of comparing the width of the cross section of the caustic surface of the beam to the width of the slab is complicated by the fact that, for complex media and/or modes that are leaky or have gain, there are not real caustic surfaces. We tried to specify the caustic by means of $|\mathrm{Re}\, x_c|$, where x_c is a (complex) value of x at which the transverse Laplacian of the field vanishes. However, disagreement between the infinitely extended medium theory and our numerical results is found even when $|\mathrm{Re}\, x_c|$ is smaller than (say, one half of) the slab half-width (Section III).

As to the energy guidance properties of the slab, they may be described by the ratio R between the power gained (from the medium) and the power lost through the boundaries by a mode while propagating over a length, say L, in the positive z-direction (Fig. 1). To this end, we consider a volume V having the same section as the slab (and unit length in the y-direction) and length L in the z-direction. If $P_S = P_{S2} + P_{S1}$ (with $P_{S1} < 0$) denotes the power flow leaving the volume V through the two bases $S2$ and $S1$, and P_W the power flow leaving the same volume through the lateral walls S_W, we have

$$R = P_S / P_W. \tag{1}$$

In the calculations, the graded index medium has been assumed to be of parabolic type, namely

$$n^2(x) = n_o^2 - n_2 \rho^2 \qquad (|\rho| \leqslant 1)$$

where n_o represent the value of n for $x=0$, and $\rho = x/d$, d denoting the half-width of the slab. The parameter n_2 has been assumed to be purely imaginary

$$n_2 = \frac{1}{2} i\epsilon \tag{2}$$

with ϵ real, so that $\mathrm{Re}\, n(x)$ turns out not to vary appreciably for $|x| \leqslant d$. Precisely, for $d = 20\lambda$, and $-2.10^{-3} \leqslant \epsilon \leqslant 2.10^{-3}$, $\mathrm{Re}\, n(d)$ varies (increases) at most by 5.10^{-8} from $\mathrm{Re}\, n_o$. Accordingly, our medium is only negligibly of the inverted type.

As to n_o, we assumed the value $n_o = 1.33 - i\ 1.10^{-5}$. Since we use the time dependence $\exp(-i\omega t)$, the sign of $\mathrm{Im}\, n_o$ indicates that the medium is active at $x=0$. The external medium (for $|x| > d$) has been assumed to be real, with $n = n_e = 1.5$. Such values are of the order of magnitude of those occurring in the technique of dye lasers [2], [3].

As stated above, the parameter ϵ has been given values from -2.10^{-3} to 2.10^{-3}. Positive values of ϵ indicate that the gain increases away from the plane $x=0$ (outwards), whereas negative values of ϵ indicate that the gain diminishes outwards (for $\epsilon < -5.10^{-5}$, the gain turns to losses somewhere inside the slab).

In such a medium, we have evaluated the complex propagation constant and the field configuration of the first four even modes as a function of ϵ. The results for the slab have then been compared with those for an infinitely extended medium and with those obtained with the Wentzel–Kramer–Brillouin (WKB) approximation [4].

II. The Complex Propagation Constant

It is well known that a planar stratified medium sustains TE and TM modes. With reference in particular to TE modes, the electric field $u(x, z)$ satisfies the scalar wave equation

$$\nabla^2 u(x,z) + k^2 n^2(x) u(x,z) = 0. \tag{3}$$

(For TM modes, see for example [5].) By putting as usual

$$u(x,z) = \psi(x) \exp(ikn_o \gamma z) \tag{4}$$

the function $\psi(x)$ is obtained by solving the reduced wave equation

$$\psi''(\rho) + k^2 d^2 \left[n^2(x) - n_o^2 \gamma^2 \right] \psi(\rho) = 0. \tag{5}$$

When $n^2(x)$ is a quadratic function of x, as in (2), the solutions of (5) are linear combinations of the parabolic cylinder functions $D_\nu(X)$, $D_\nu(-X)$, where (6), (7)

$$\nu = -\frac{1}{2} + k^2 d^2 n_o^2 \frac{1-\gamma^2}{\alpha^2}$$

$$X = \alpha\rho \tag{6}$$

and α indicates any one of the roots of

$$\alpha^4 = 4k^2 d^2 n_2. \tag{7}$$

Even modes are simply given by

$$\psi(\rho) = D_\nu(X) + D_\nu(-X). \tag{8}$$

The eigenvalue γ appearing in the generally complex parameter ν is to be determined by imposing the boundary conditions at $x=d$ to the field (8), which yields

$$\alpha \frac{D_\nu'(\bar{X}) - D_\nu'(-\bar{X})}{D_\nu(\bar{X}) + D_\nu(-\bar{X})} = ikd\left(n_e^2 - n_o^2\gamma^2\right)^{1/2} \tag{9}$$

where $\bar{X} = X(x=d) = \alpha$ and $\mathrm{Re}(n_e^2 - n_o^2\gamma^2)^{1/2} \geqslant 0$. The right-hand side of (9) derives from the fact that the field in the outer medium is an outgoing plane wave, with the same z-dependence $\exp(ikn_o\gamma z)$ as $u(x,z)$, namely

$$u^e(x,z) = A \exp\left[ik(n_e \alpha_e x + n_o \gamma z)\right] \qquad (\text{for } x\ d). \tag{10}$$

In the infinitely extended medium (no-boundary case), the complex propagation constant is simply given by [1]

$$n_o \gamma = \left[n_o^2 - \frac{2m+1}{kd} n_2^{1/2} \right]^{1/2} \tag{11}$$

with $\mathrm{Re}\, n_2^{1/2} > 0$, and m is even for even modes.

Equation (9) has been solved numerically, with the help of an electronic computer (Eclipse Data General). The

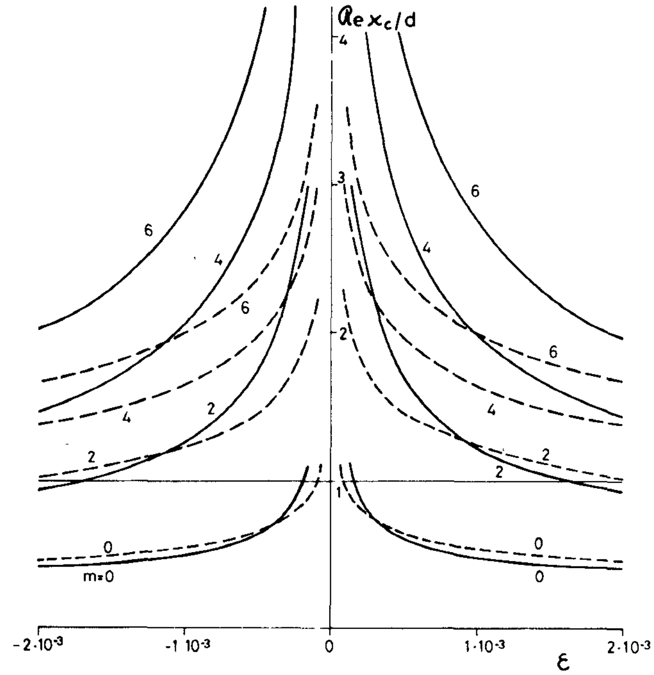

Fig. 2. $|\mathrm{Re}\, x_c|/d$ plotted versus ϵ for the first four even modes ($m=0,2,3,4$). Dashed lines refer to the no-boundary theory.

results of the computations are shown in the following Figs. 2–7.

Fig. 2 shows $|\mathrm{Re}\, x_c|/d$ plotted versus ϵ (solid lines). Dashed lines represent the same quantity as deduced from the no-boundary treatment (11). It appears from the figure that for one and the same mode, the approximate values tend (from above) to the exact ones when $|\epsilon|$ increases, whereas when $|\epsilon|$ is small enough, the approximate values are closer to $x=0$ than the exact ones. Thus, if in some way $\mathrm{Re}\, x_c$ specifies the caustic, it turns out that for $|\epsilon|$ small enough, the no-boundary caustic is closer to the plane $x=0$ than the exact one. Moreover, it appears that, in the numerical cases examined, the mode $m=0$ has the caustic internal to the slab for $|\epsilon|>2.10^{-4}$, while the caustic of the mode $m=2$ enters the slab at $|\epsilon|\simeq 1.7\ 10^{-3}$. It may be noted that the position of the caustic is substantially independent of the sign of ϵ, namely of the sense of $\mathrm{Im}\, n(x)$, whether outwards or inwards.

Fig. 3 shows $\mathrm{Re}\, n_o\gamma$ plotted versus m for several values of ϵ. Dashed lines refer to the no-boundary case. It appears that, for the no-boundary case, $\mathrm{Re}\, n_o\gamma$ depends appreciably on ϵ, contrarily to what happens with the solutions of (9). It may be worth noting that the solid line is well fitted by the function

$$\mathrm{Re}\, n_o\gamma = \mathrm{Re}\left[n_o - \frac{(m+1)^2}{32\eta^2 n_o} \right] \tag{12}$$

where $\eta = d/\lambda$, which expression may be deduced by assuming the slab to be made of a homogeneous medium with refractive index $\mathrm{Re}\, n_o$.

Fig. 4 shows $\mathrm{Im}\, n_o\gamma$ plotted vs ϵ for the first four even modes (solid lines). Dashed lines represent the values of $\mathrm{Im}\, n_o\gamma$ deduced from the no-boundary treatment, and are apparently inadequate to be applied to the slab—a result already found in the numerical cases examined in [4]. Solid lines are unexpectedly straight lines over the entire considered range of ϵ. It is worth noting that the straight line labelled $m=0$ is in a good agreement, for $|\epsilon|$ larger than

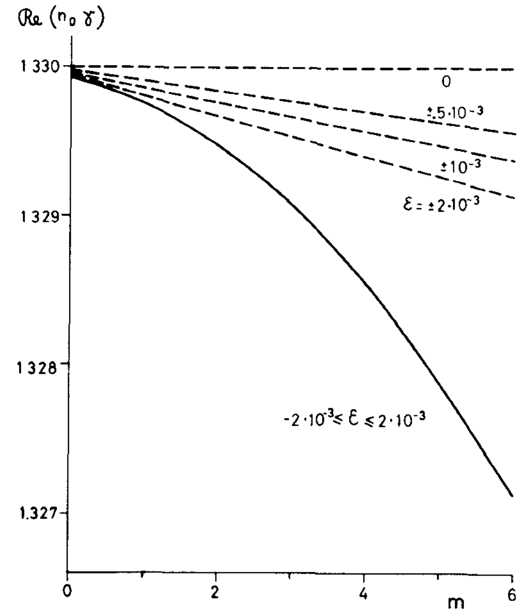

Fig. 3. $\mathrm{Re}(n_o\gamma)$ plotted versus m for several values of ϵ. Dashed lines refer to the no-boundary theory.

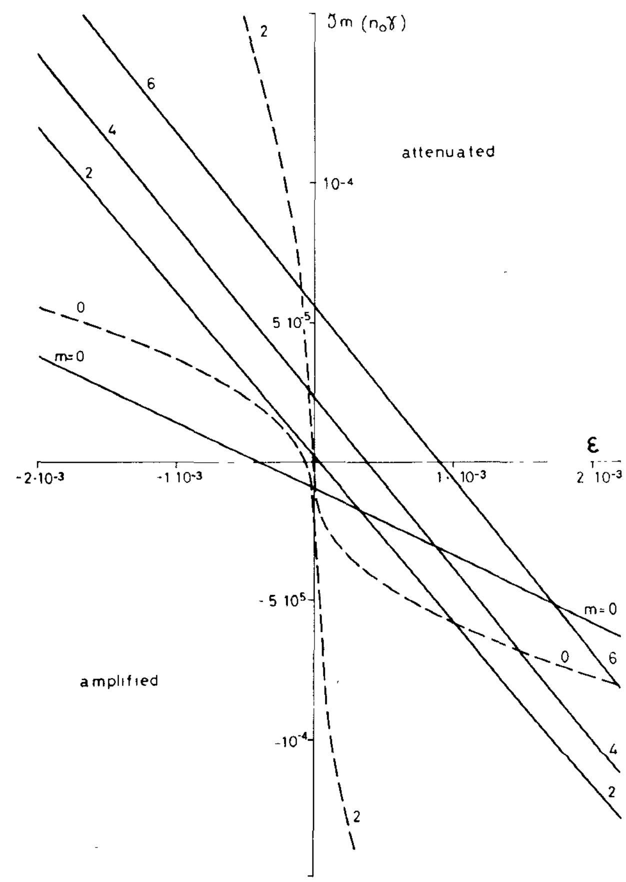

Fig. 4. $\mathrm{Im}(n_o\gamma)$ plotted versus ϵ. Dashed lines refer to the no-boundary theory for the first two even modes only.

$\simeq 10^{-3}$, with the values deduced from the WKB treatment, in the form valid when the caustic is inside the slab (see Appendix). Conversely, the slope common to the straight lines corresponding to the higher order modes may be deduced from the WKB treatment in the form valid for the cases when the caustic is out of the slab (in other words, when $|\epsilon|$ is small enough). This yields

$$\operatorname{Im} n_o\gamma = \operatorname{Im} n_o + \operatorname{Re}\left[\frac{(m+1)^2}{32\pi\eta^3 n_o \Delta n} - \frac{\epsilon}{12 n_o}\right] \tag{13}$$

where $\Delta n = (n_e^2 - n_o^2)^{1/2}$.

Fig. 4, and the comparison with the WKB treatment, seem to indicate that the fundamental mode behaves differently from the higher order modes. Roughly speaking, we would say that the higher order modes are controlled by the dielectric discontinuity $n_o - n_e$ as in conventional homogeneous thin-film guiding, whereas the fundamental mode is controlled both by the discontinuity at the boundary and by the continuous inhomogeneity of the medium. This however does not mean that the fundamental mode is a "guided" mode in all cases, in the sense specified in the Introduction. It is easily shown that a mode is leaky, and therefore its amplitude increases exponentially away from the slab, when $\operatorname{Im} n_o\gamma > 0$, while a mode is guided, and its energy peaks up inside the slab and decreases exponentially away from it, when $\operatorname{Im} n_o\gamma < 0$. This follows simply, with reference to (10), by noting that

$$(n_e\alpha_e)^2 + (n_o\gamma)^2 = n_e^2. \tag{14}$$

Clearly, if n_e is real, as we assumed, $\operatorname{Im} n_e\alpha_e$ has the sign opposite to that of $\operatorname{Im} n_o\gamma$. As a consequence, we have that a mode is of guided type, if its gains in the medium enough to compensate for the radiation losses through the lateral walls, since in this case $\operatorname{Im} n_o\gamma < 0$. Otherwise, if the losses through the walls exceed the power provided by the medium, $\operatorname{Im} n_o\gamma > 0$ and the mode is leaky.

The same conclusion (namely that a mode is leaky or guided according as $\operatorname{Im} n_o\gamma$ is positive or negative) does not hold for the modes derived from the no-boundary theory, since for them (14) cannot apply. A mode whose amplitude decreases away from a region near the plane $x=0$ is always found in the infinitely extended medium [1], and is "stable" too, for $\epsilon<0$.

III. Mode Configurations and Power Flow

Once the propagation constant $kn_o\gamma$ is determined for the various modes, (8) with ν and α given by (6) and (7) allows one to evaluate the transverse distribution $\psi(\rho)$ of the field. Fig. 5 shows the amplitude $|\psi(\rho)|$, normalized to 1 at $\rho=0$, for the first four modes we have examined. The dashed lines indicate the slight variations of $|\psi(\rho)|$ dependent on ϵ. It appears that the amplitude configurations are largely independent of ϵ, and in particular of its sign.

The phase distributions of the two modes $m=0$ and $m=2$ are shown in Fig. 6 for $|\epsilon| = 2.10^{-3}$. From such a figure and from Fig. 5 it turns out that changing ϵ into $-\epsilon$ changes a particular mode into one which is approximately

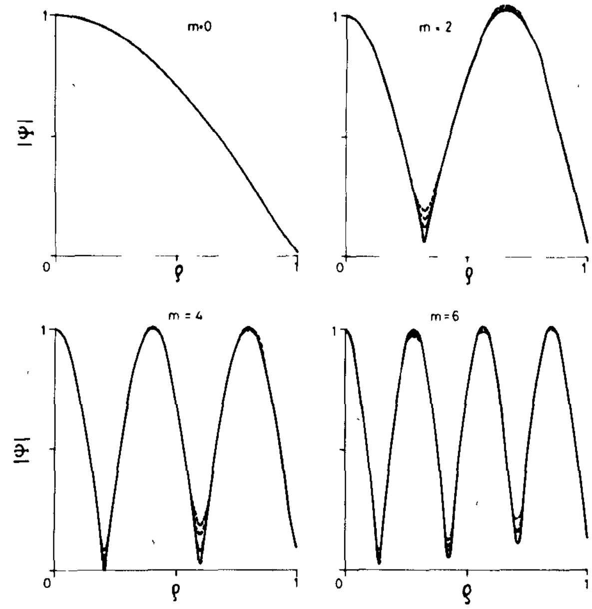

Fig. 5. Amplitude distributions $|\psi(\rho)|$ of the first four even modes.

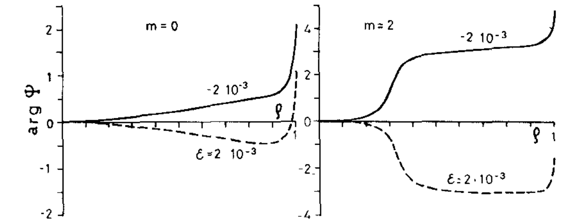

Fig. 6. Phase distributions $\arg\psi(\rho)$ of the first two even modes ($m=0,2$).

the complex conjugate of the original, almost everywhere in the range $-d \leqslant x \leqslant d$. Qualitatively, this is expected, since $n^2(x,-\epsilon)$ is almost the complex conjugate of $n^2(x,\epsilon)$. However, at $x \simeq d$, the phase has a jump of about $+\frac{1}{2}\pi$ in both cases. This distortion of the wavefronts near the boundaries is not predicted by the no-boundary treatment, but it is physically necessary, in the case $\epsilon>0$, for the energy to flow towards the exterior of the slab. Accordingly, it is expected that the presence of the refractive index discontinuity at the walls of the slab will affect the field inside the slab in all cases when the gain increases outwards or the losses decrease outwards, independently of how small the field is at the walls.

Fig. 7 shows the ratio R defined by (1), plotted versus ϵ. With reference to a portion of slab of length L, extending from z to $z+L$, we can write

$$P_S = \frac{kd}{\omega\mu}\operatorname{Re}(n_o\gamma)\left[e^{-2kL\operatorname{Im}(n_o\gamma)} - 1\right] \cdot e^{-2kz\operatorname{Im}(n_o\gamma)} \int_0^1 \psi(\rho)\psi^*(\rho)\,d\rho$$

$$P_W = -\frac{1}{2\omega\mu}\frac{\operatorname{Re}(n_e\alpha_e)}{\operatorname{Im}(n_o\gamma)}\left[e^{-2kL\operatorname{Im}(n_o\gamma)} - 1\right] \cdot e^{-2kz\operatorname{Im}(n_o\gamma)}\psi(1)\psi^*(1).$$

Hence

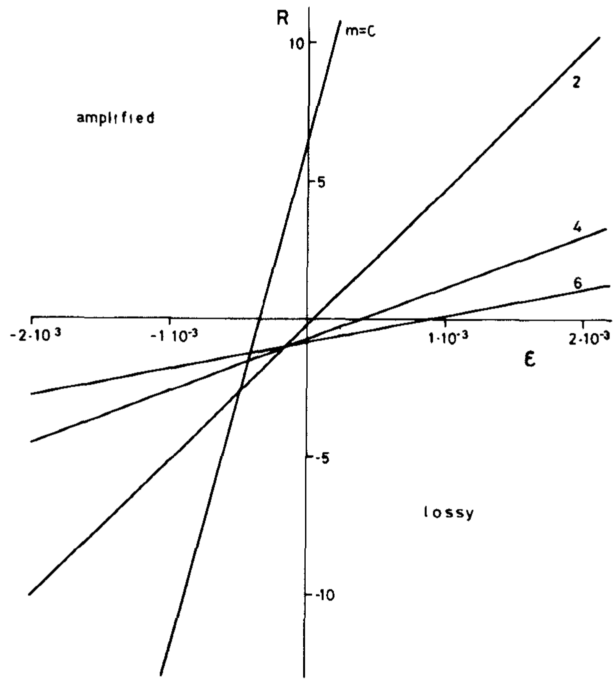

Fig. 7. The power ratio R, defined by (1), plotted versus ϵ for the first four even modes.

$$R=\frac{P_S}{P_W}=-2kd\frac{\mathrm{Re}(n_o\gamma)\mathrm{Im}(n_o\gamma)}{\mathrm{Re}(n_e\alpha_e)}\left[\psi(1)\psi^*(1)\right]^{-1}\cdot\int_0^1\psi(\rho)\psi^*(\rho)d\rho. \qquad (15)$$

The linear trend of all modes in Fig. 7 is due to the fact that, in the considered range of ϵ, both $\mathrm{Re}(n_o\gamma)$ and $|\psi(\rho)|$ are practically independent of ϵ, as appears from Figs. 3 and 5, respectively, $\mathrm{Im}(n_o\gamma)$ is a linear function of ϵ for all modes (see Fig. 4), and $\mathrm{Re}(n_e\alpha_e)$ turns out to be almost independent of ϵ, according to the relation

$$\mathrm{Re}(n_e\alpha_e)=\mathrm{Re}\left(n_e^2-n_o^2\gamma^2\right)^{1/2}\simeq\left[n_e^2-(\mathrm{Re}\,n_o)^2\right]^{1/2}. \qquad (16)$$

IV. Conclusion

In the present paper we have investigated the mode and energy guidance properties of a slab of graded index medium where there are transverse variations of $\mathrm{Im}\,n(x)$ only.

The new, slightly surprising findings are that: 1) the mode configurations and the real part of the propagation constant turn out to be practically independent of the parameter ϵ describing the grad Im $n(x)$; 2) the imaginary part of $n_o\gamma$ and the power ratio R between the power gained and the power lost through the boundaries turn out to be linear functions of ϵ.

This occurs over a range of values of ϵ sufficiently large (about two orders of magnitude larger than the values achievable in flashlamp pumped dye lasers) to let the imaginary part of the propagation constants pass from positive values (leaky modes) to negative values (amplified modes).

For the higher order modes these behaviors are described with a good agreement by the WKB approximation in the form it takes for small $|\epsilon|$. For the fundamental mode, however, good results are derived from the WKB approximation, in the form valid when the caustic is inside the slab, only for values of ϵ not too small.

The physical interpretation of these results seems to be that the presence of the boundaries affects substantially the modal propagation even when the field is very small at the boundaries, or, in other words, when the gain (loss) inhomogeneities are so large to shift the caustic of the mode into the central region of the slab. This conclusion is also supported by the fact that the wavefronts near the boundaries have to present in any case the curvature of a diverging beam, as noted in connection with Fig. 6.

Another result which may be worth noting in Fig. 4 is that the gain of the low order modes (in our case of the fundamental mode) may be smaller than the gain of some higher order modes. This happens when ϵ is sufficiently large, namely when the gain near the boundaries of the slab is sufficiently larger than the gain in the central region. In these conditions, the low order modes which are confined in the low-gain region (Fig. 2) may gain less than the higher-order modes, in spite of the fact that the last ones suffer larger radiation losses through the boundaries. This phenomenon has been known for a long time.

Appendix

The numerical results reported in Figs. 3 and 4 have been compared with those derivable from the WKB approximation, by using the formulas of [4], and by generalizing them to include all cases of interest in the ranges of parameters here examined. Such formulas are as follows:

When $\mathrm{Re}\,\rho_c\gg 1$, where $\rho_c=x_c/d=n_o[(1-\gamma^2)n_2]^{1/2}$, the field $\psi(\rho)$ of even modes can be written as

$$\psi(\rho)=\left[S'(\rho)\right]^{-1/2}\cos\left[kdS(\rho)\right] \qquad (A.1)$$

where $\rho=x/d$, $S'=dS/d\rho$ and

$$S(\rho)=\int_0^\rho\left[n^2(\rho)-n_o^2\gamma^2\right]^{1/2}d\rho. \qquad (A.2)$$

When $\mathrm{Re}\,\rho_c\simeq 1$, one has

$$\psi(\rho)=\bar{a}\,\mathrm{Ai}(-Y)+\bar{b}\,\mathrm{Bi}(-Y) \qquad (A.3)$$

where Ai and Bi denote the Airy functions [7] and

$$Y=\left(k^2d^2f\right)^{1/3}(\rho_c-\rho)$$
$$f=-\frac{d}{d\rho}n^2(\rho)\Big|_{\rho=\rho_c}$$
$$\bar{a}=\left(kd/f\right)^{1/6}\pi^{1/2}\cos\left(\phi-\frac{\pi}{4}\right)$$
$$\bar{b}=\left(kd/f\right)^{1/6}\pi^{1/2}\cos\left(\phi+\frac{\pi}{4}\right)$$
$$\phi=kdS(\rho_c). \qquad (A.4)$$

Finally, if $\mathrm{Re}\,\rho_c\ll 1$, one has

$$\psi(\rho)=\left[S'(\rho)\right]^{-1/2}\left[a'e^{ikdS(\rho)}+b'e^{-ikdS(\rho)}\right] \qquad (A.5)$$

where

$$a' = 2^{-\delta} e^{-i(\phi \mp \pi/4)} \cos(\phi \pm \pi/4)$$

$$b' = 2^{\delta - 1} e^{i(\phi \pm \pi/4)} \cos(\phi \mp \pi/4). \qquad \text{(A.6)}$$

In (A.6), $\delta = 0$ or $\delta = 1$ according as $\operatorname{Im} n_2 > 0$ or $\operatorname{Im} n_2 < 0$, and the upper (lower) signs hold for $\operatorname{Im} n_2 > 0$ ($\operatorname{Im} n_2 < 0$). In [4], only the formulas valid for $\operatorname{Im} n_2 > 0$ were reported.

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Short Papers

Two Simple Methods for the Measurement of the Dielectric Permittivity of Low-Loss Microstrip Substrates

R. M. PANNELL AND B. W. JERVIS

***Abstract*—Two simple methods are presented for the measurement of the dielectric permittivity of low-loss microstrip substrates. The permittivity associated with a specific length of microstrip may be obtained. The methods are not wasteful of substrate material.**

I. Introduction

During an investigation of attenuation in microstrip transmission lines [1] built using cheap substrate materials, it became necessary to measure the dielectric permittivity of the substrate beneath the top conductor in order to design for the required characteristic impedance. Two simple methods of doing this were invented and will be referred to as the Reflection Cancellation Method and the Line Balancing Method.

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The characteristic impedance Z_0 (Ω) of a microstrip line is partly determined by the dielectric permittivity ϵ_r of the substrate material [2]. Further, because microstrip is a mixed-dielectric transmission line it exhibits an effective dielectric permittivity, $\epsilon_{r\text{eff}}$, which is less than ϵ_r [2]. At high frequencies dispersion becomes significant and the frequency-dependent permittivity, $\epsilon_{r\text{eff}}(f)$ given by [3]

$$\epsilon_{r\,\text{eff}}(f) = \epsilon_r - \frac{\epsilon_r - \epsilon_{r\,\text{eff}}}{1 + G(f/f_p)^2} \qquad (1)$$

where $G = 0.6 + 0.009 Z_0$ and $f_p = 10^7 Z_0 / 8\pi h$ must be used.

Relatively simple techniques for measuring the average permittivity of a complete microstrip substrate have appeared in the literature. Some have involved producing special resonant structures [4]–[7] and are wasteful of possibly expensive substrate material. Less wasteful approaches [8]–[10] have involved determining the permittivity from the measured cavity resonance frequencies of the double metal-clad substrate prior to etching.

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